2.2 Introductions to Proofs

- Theorem → a statement that can be proven to be true
- Axioms → statements assumed to be true

- Before doing a proof write "proof"
- "■" end of proof symbol

- "The sum of two positive real numbers is strictly greater than the average of the two numbers."
  - $\forall x \forall y [(x>0 \text{ and } y>0) \rightarrow x+y > \frac{x+y}{2}]$

- "There is an integer that is equal to its square."
  - $\exists x (x = x^2)$

- Proof by <u>exhaustion</u> (used for universal statements)
  - Usually used when domain is very small
  - prove the statement by checking each element in the domain individually.

- Proof by <u>universal generalization</u> (used for universal statements)
  - Uses an arbitrary element in the domain and proves the statement for that element

- <u>Counterexample</u> → an assignment of values to variables that shows that a universal statement is false

- Counterexample for conditional statements → must satisfy all the hypotheses and contradict the conclusion

- existance proof → A proof that shows an existential statement is true

- A constructive proof of existance gives a specific example of an element in the domain or a set of directions to construct an element in the domain that has the required properties.

- A nonconstructive proof of existance proves that an element with the required properties exists without giving a specific example.

Disproving existential statements

- To disprove "There is a real number whose square is negative"
  - You have to prove "The square of every real number is greater than or equal to 0."

2.3 Best Practices and common errors in proofs

- Allowed assumptions in proofs:
  - The rules of algebra
  - The set of integers is closed under addition, multiplication, and subtraction.
  - Every integer is either even or odd
  - If x is an integer, there is no integer between x and $x+1$
  - The relative order of any two numbers ($\frac{1}{2} < 1$ or $4.2 \geq 3.7$)
  - The square of any real number is greater than or equal to 0.

The language of Proofs:

- "Thus" and "Therefore"
  - A statement that follows from the previous statement or previous few statements can be started with "Thus" or "Therefore"
- "Let"
  - New variable names are often introduced with the word "let"
- "Suppose"
  - used to introduce a new variable or introduce a new assumption
- "Since"
  - Example: "Since $x > 0$ and $y > z$ then $xy > xz$"
- "We will prove" and "We will show"
  - Starting the proof
- "By definition"
  - Use before stating a fact that is known because of a definition.
- "By assumption"
  - use before stating a fact that is known because of an assumption
- "In other words"
  - used when rephrasing a statement in a more useful way
- "gives" and "yields"
  - Example: "Substituting $m = 2k$ into m^2 yields $(2k)^2$"

Best Practices in writing proofs

1. Indicate start and end of proof.
   - Start with "Proof:", and end with ■
2. Write proofs in complete sentences
3. Give reader a roadmap of what has been shown, what is assumed, and where the proof is going.
4. Introduce each variable when the variable is used for the first time.
5. A block of equations should be introduced with English text and each step that does not follow from algebra should be justified.

Existential instantiation

- A law of logic that says if an object is known to exist, then that object can be given a name as long as the name is not being used for something else.

2.4 Writing Direct Proofs

- In a <u>direct proof</u> of the conditional statement "$p \to q$", the hypothesis p is assumed to be true and the conclusion q is proven as a direct result of the assumption.

- "For every integer n, if n is odd then n^2 is odd." $D(n) =$ "n is odd"
  - $\forall n \, (D(n) \to D(n^2))$
    - * the proof starts with n, an arbitrary integer
    - * assumes n is odd
    - * then proves that n^2 is odd

Example

Theorem: The square of every odd integer is also odd.

Proof: let n be an odd integer

Since n is odd, $n = 2k+1$, for some integer k

Plug in $n = 2k+1$ into n^2 to get:

$$n^2 = (2k+1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$

Since k is an integer, $2k^2 + 2k$ is also an integer

Since $n^2 = 2m + 1$, where

$m = 2k^2 + 2k$ is an integer,

n^2 is odd $\blacksquare$

2.5 Proof by contrapositive

- A proof by contrapositive proves a conditional theorem $p \to q$ by showing that the contrapositive $\neg q \to \neg p$, where $\neg q$ is assumed to be true.

- "For every integer n, if n^2 is odd then n is odd." $\leftrightarrow$ $\forall n (D(n^2) \to D(n))$ $D(n) = n$ is odd
    - To prove by contrapositive you would:
        * start with arbitrary integer n
        * assume $\neg D(n)$ is true
        * then prove $\neg D(n^2)$ is true

When to use direct proof vs a proof by contrapositive

- Decide whether the hypothesis or the negation of the conclusion provides a more useful assumption to work with.

Example

What is the starting assumption and what is proven in a proof by contrapositive of this statement: "if x and y are integers such that xy is even, then x is even or y is even."

* Starting assumption: x is odd and y is odd
* what is proven: xy is odd

Proofs by contrapositive of conditional statements with multiple hypotheses

- In a proof by contrapositive it is only necessary to show that one of the hypotheses is false.

because:

- if H_1 and H_2 are both true then C is true $(H_1 \wedge H_2) \rightarrow C$

Contrapositive:

- if C is false, then it cannot be the case that H_1 and H_2 are both true: $\neg C \rightarrow \neg(H_1 \wedge H_2)$

DeMorgans Law:

- if C is false, then H_1 is false or H_2 is false: $\neg C \rightarrow (\neg H_1 \vee \neg H_2)$

which in turn is equivalent to:

- if C is false and H_1 is true, then H_2 is false: $(\neg C \wedge H_1) \rightarrow \neg H_2$